Chapter 15
Modeling Techniques in Forming Processes


THE OBJECTIVE OF MANUFACTURING is the production of a consistent quality product at a minimal cost. Generally effective goals include shortening the lead time in the design cycle, reducing tooling cost and machine downtime at the production stage, and developing a stable process with a minimal reject rate. The ability to predict the performance of a particular manufacturing process and to compare it with alternative manufacturing processes at an early stage in the process design cycle furthers these goals by reducing costly trial-and-error design iterations using production equipment.

Numerous analytical techniques have been developed to improve the process designer’s ability to evaluate a process and to predict various aspects of the metal forming process. Early methods relied on simple analytical techniques, such as the slab method, the slip-line method, the upper bound method, and heuristics to predict forming load, critical ratios, workability limits, and die design features. Complex analytical equations were converted to charts, which could be applied by the designer.

As computer technology became prevalent in the engineering and manufacturing industry, analytical techniques, such as the upper bound method, were used to develop specialized computer programs, which could be used to analyze a particular process, such as tube sinking, strip rolling, or extrusion. The finite element method (FEM) for metal forming applications was first introduced in early 1970 (Ref 1). The continuous improvement in computer technology and FEM has made an important impact in the metal forming industry in the mid-1980s. Due to its unique capability in describing complex shapes, boundary conditions, and realistic material thermomechanical response, the development of a general-purpose metal forming analysis software has been realized. The method has been used as an essential tool for product and process design engineers to reduce development time and cost. Due to the demand from the industry to produce more accurate simulation models, the FEM has continuously evolved from two-dimensional analysis into the true three-dimensional models since the late 1980s and early 1990s (Ref 2–4).

This chapter reviews the overall development of modeling techniques for forming processes, including:

- Slab method
- Slip-line method
- Upper bound method
- Finite element method

Modeling Techniques

The most fundamental calculations used in metal forming analysis involve a forming load estimate, which is useful in selecting the size of equipment required to form the product. The simplest formula takes the form:

\[ P = K \sigma A \]  \hspace{1cm} (Eq 1)

where \( P \) is the forming load, \( \sigma \) is the mean flow stress of a workpiece material under an idealized state of deformation, \( A \) is the planar area of the workpiece, and \( K \) is an empirically determined correction factor for a particular forming process. The correction factor reflects the effects of nonuniformity of deformation and friction between the workpiece and tool.

This estimation has been improved upon by more elaborated approximate solution techniques such as the slab method, the slip-line method, the upper bound and lower bound methods, and the finite element methods. These methods have been employed to allow for the estimation of nonuniform deformation and friction between the workpiece and tool.

Figure 1 illustrates a simple example of the slab method applied to a ring compression problem. In fact, the ring compression test has been used widely as a convenient tool to evaluate the friction factor between the tool and workpiece. In bulk metal forming industries, the constant shear friction model (based on shear strength of the workpiece) has been preferred to the well-known Coulomb friction model (based on contact pressure). In Fig. 1(a), the inner radius \( R_i \), the outer radius \( R_o \), and the height \( H \) characterize the geometry of the ring. Now consider an infinitesimal element depicted in Fig. 1(b). The equilibrium equation in the \( \phi \) direction can be written as follows:

\[ \sigma_r H d\phi - (\sigma_\phi + \sigma_r) H (r + dr) d\phi + 2\sigma_\phi \sin \left( \frac{d\phi}{2} \right) H dr + 2 \mu kr d\phi = 0 \]  \hspace{1cm} (Eq 2)
where $m$ and $k$ are the shear friction factor and the equivalent shear stress, respectively, and the sign $\pm$ is used to express that the direction of frictional stress changes from outward to inward at the neutral point as $r$ increases. The neutral point is denoted by the radius $R_n$. Assuming that $\sigma_r = \sigma_n$ and neglecting higher order terms, Eq 2 can be simplified as:

$$\sigma_r = Y + \sigma_n$$

(Eq 8)

where $Y$ is the yield stress of a material. It is noted that $Y = \sqrt{3} k$ for a material complying with von Mises yield criterion. The forming load can then be evaluated as:

$$P = \int_{R_n}^{R} 2\pi r \sigma_r dr + \int_{R_n}^{R} 2\pi r \sigma_n dr$$

$$= K_{Slab} \sigma_A A$$

(Eq 9)

where $K_{Slab}$ is the correction factor or normalized forming load defined by:

$$K_{Slab} = 1 + \frac{1}{2} \frac{m}{\sqrt{3}} \left( \frac{R}{R_n} - 1 \right)$$

(Eq 10)

The slip-line method is another simple and powerful classical solution method, although its application is limited to plane-strain problems for a rigid-plastically deforming material (Ref 5, 6). In this method, the equilibrium equations for a plane-strain state are first transformed into the hyperbolic differential equations expressed in terms of the mean stress, the maximum/minimum shear stress, and the direction of maximum/minimum shear stress. The characteristics of the hyperbolic differential equations are known as the slip-lines. The slip-line field then can be constructed by networking two kinds of slip-lines representing the maximum and minimum constant shear lines that are orthogonal to each other. Several useful techniques have been proposed to construct the slip-line field graphically depending on the configuration of a problem and the associated boundary conditions. The forming load can then be obtained by determining integral constants for particular slip lines from the known state of stresses at some points. The slip-line method has also been applied successfully to various plane-strain forming problems such as indentation, extrusion, drawing, and rolling.

**Upper-Bound Method.** Unlike the two previously discussed methods, the upper-bound method (UBM) is based on the energy principle, known as the upper-bound theorem. The upper-bound theorem states that the rate of total energy associated with any kinematically admissible velocity field defines an upper bound to the actual rate of total energy required for the deformation. Hence, for a given class of kinematically admissible velocity fields, the velocity field that minimizes the rate of total energy is the lowest upper bound, and therefore is nearest the actual solution. Here, the kinematically admissible velocity field is used to denote a velocity field that satisfies the incompressibility requirement for a rigid-plastic material and the prescribed velocity boundary conditions. However, the velocity field may be discontinuous on a finite number of imaginary internal surfaces. The rate of total energy generally consists of three terms such that:

$$E_T = \int_V \bar{\sigma} \bar{\varepsilon} dV + \int_{S_D} k \bar{\varepsilon}_D dS + \int_{S_k} m \bar{D}_k dS$$

(Eq 11)

where $\bar{\sigma}$ and $\bar{\varepsilon}$ are the equivalent stress and the equivalent strain rate, respectively; $\Delta V_k$ is the magnitude of velocity discontinuity tangent to the velocity discontinuity surfaces $S_k$; and $\Delta V_2$ is the magnitude of sliding velocity on the contact surface $S_p$.

Each term in the right-hand side of Eq 11 represents the rate of plastic deformation energy, the rate of energy dissipation associated with internal velocity discontinuity, and the rate of energy dissipation due to friction between the tool and workpiece, respectively. The second term, also known as the *jump condition*, can be omitted when a class of continuous velocity fields is considered. Among various classical solution methods, the upper-bound method has been applied most extensively to various two-dimensional or three-dimensional forming problems because it delivers a fast and accurate solution as long as the trial velocity field can be provided closer to the actual velocity field. However, it is not easy to choose a good trial velocity field using a combination of analytic functions for geometrically complicated problems. In order to relax in such a difficulty, the upper-bound elemental technique (UBET), based on the concept of a "unit rectangular deforming region" (Ref 10) has been developed and applied to rather complex forming problems and preform design applications.

In order to compare the characteristics of solutions with different solution methods, i.e., the upper-bound and slab methods, the ring compression problem illustrated in Fig. 1 is employed as an example here. Also, a simple form of the trial velocity field is chosen so that an explicit form of solution can be obtained. The trial
velocity field used in the present example has the form (Ref 11):

\[ u(r) = \frac{D}{2} \left( r - \frac{r_0^2}{r} \right) \]

\[ v = 0 \]

\[ w(z) = -Dz \]  

(Eq 12)

where \( D = \frac{V_D}{H} \), \( V_D \) is the die velocity, \( H \) is the height of ring, and \( r_0 \) is the radius at a neutral point. It is noted that this velocity field contains only one unknown, i.e., \( r_0 \), and it is adequate for investigating the effect of the location of neutral point on the forming load. The neutral radius \( r_0 \) is defined by:

\[ u = 0 \text{ at } r = r_0 \]

\[ u < 0 \text{ for } r < r_0 \]

\[ u > 0 \text{ for } r > r_0 \]  

(Eq 13)

Because the radial component of velocity, \( u \), is a function of \( r \) only, the present trial velocity field cannot reproduce the so-called barreling or bulging phenomenon due to friction.

All nonzero strain-rate components are then written as:

\[ \dot{e}_r = \frac{D}{2} \left( 1 + \frac{r_0^2}{r^2} \right) \]

\[ \dot{e}_\theta = \frac{D}{2} \left( \frac{r_0^2}{r^2} - 1 \right) \]

\[ \dot{e}_z = -D \]  

(Eq 14)

It is noted that the velocity field satisfies the incompressibility condition such that:

\[ \dot{e}_r + \dot{e}_\theta + \dot{e}_z = 0 \]  

(Eq 15)

Since the present velocity field satisfies the continuity requirement, the incompressibility condition, and the prescribed velocity boundary condition, it is proved to belong to a class of kinematically admissible velocity fields. It is also noted that the second term in Eq 11 can be omitted because there is no internal velocity discontinuity.

The rate of total energy in Eq 11 can be rewritten as:

\[ \dot{E}_T = \oint \sigma : \dot{e} \, dV + \int \dot{m} \dot{\nu}_i \, dS \]

\[ = \int_R \sigma : \dot{e} (2\pi H) \, dr + 2 \int_R \dot{m} \dot{\nu}_i (2\pi r) \, dr \]  

(Eq 16)

After substituting the following expression of the equivalent strain-rate:

\[ \dot{\varepsilon} = D \left( 1 + \frac{1}{3} \left( \frac{r_0}{r} \right)^2 \right) \]  

(Eq 17)

and the radial velocity \( u \) defined in Eq 12 into Eq 16, the rate of total energy \( \dot{E}_T \) can be integrated explicitly as:

\[ \frac{\dot{E}_T}{2 \pi k V_D} = \frac{1}{2} \left( \sqrt{r_0^4 + 3R_i^4} - \sqrt{r_0^4 + 3R_i^4} \right) \]

\[ -\frac{r_0^2}{2} \left( \log \left( \frac{r_0 + \sqrt{r_0^4 + 3R_i^4}}{r_0 + \sqrt{r_0^4 + 3R_i^4}} \right) - \log \left( \frac{R_i}{R_i} \right) \right)^2 \]

\[ + m \left( \frac{4}{3} \frac{r_0^2}{R_i^2} - (R_i + R_s) \right) \]  

(Eq 18)

The unknown coefficient, i.e., the neutral radius \( r_0 \), is then determined so that the rate of total energy attains to its minimum value such that:

\[ \frac{d\dot{E}_T}{dr_0} = 0 \]  

(Eq 19)

After some manipulation, the neutral radius can be written as:

\[ r_0 = \frac{1}{2} \left( R_i + R_s \right) + \frac{1}{4} \frac{H}{m} \left( \frac{\beta_i}{R_i} + \frac{\beta_s}{R_s} \right) \]  

(Eq 20)

where \( \beta_i = \frac{\alpha_i}{R_i} \) and \( \beta_s = \frac{\alpha_s}{R_s} \). Equation 20 can then be solved for \( r_0 \) numerically with specific values of the ring geometry, i.e., \( R_i, R_s, \) and \( H \), and the shear friction factor \( m \).

The forming load can be evaluated directly by substituting the value of neutral radius into Eq 18 and with use of the relation:

\[ \frac{P}{\dot{E}_T V_D} = \frac{1}{2} \left( R_i + R_s \right) \]  

(Eq 21)

The correction factor or normalized forming load \( K_{UBM} \) can be defined by:

\[ K_{UBM} = \frac{P}{\dot{E}_T V_D} \]  

(Eq 22)

**Finite Element Method**

Metal forming simulation is often classified as a class of highly nonlinear continuum mechanics problems because it is accompanied by large deformation (geometric nonlinearity), nonlinear materials behavior (material nonlinearity in both deformation and temperature), and frictional contact (nonlinear boundary condition). Starting from the mid-1980s, the FEM has shown great success in the axisymmetric applications such as disk forging, cold forging of cylindrical fasteners, and so forth. The approximation of a two-dimensional cross section in a three-dimensional part, using the plane-strain assumption, is an alternative to achieving some understanding of the three-dimensional forming process. Although a considerable amount of research has been done in developing the FEM for metal forming simulation since the pioneering work (Ref 1) was presented in 1973, rigorous three-dimensional simulation of metal forming problems still remains a challenging task from the standpoint of computational efficiency, solution accuracy, graphics visualization, mesh generation and automatic remeshing, and so on. As computer technology and FEM advance, wider and more complicated metal forming processes are being investigated. It is believed that the further development of FEM will be continuously challenged by the need from the industry to make the modeling more accurate, more practical, and more affordable.

Since the two-dimensional FEM implementation has been discussed elsewhere in detail (Ref 12), this section focuses on the three-dimensional implementation.

**Preliminary Assumptions**

In order to narrow down the discussion to the most practical applications among a variety of metal forming simulations, the following preliminary assumptions are first introduced.

**Quasi-Static Analysis.** In most metal forming processes, dynamic effects can be neglected except for high strain-rate processes in which a realistic deformation mode cannot be obtained without considering the effect of stress wave propagation and in which the magnitude of kinetic energy is comparable to that of deformation energy. In cases where dynamic effects can be neglected, the progress of deformation is analyzed in a manner such that every instantaneous state of a body in the course of deformation is satisfied with the equilibrium conditions. Such a method of analysis is generally called the quasi-static analysis.

It is noteworthy that even for forming processes between medium and low strain-rate ranges, the explicit method of solution originally designed for dynamic analyses is sometimes used for the sake of computational efficiency. For the explicit method, however, the size of time step must be small enough, typically of the order of microseconds, to satisfy the stability criterion of the explicit time marching scheme. It means that the computational efficiency may not be achieved because the number of solution steps increases greatly compared with that of the implicit method. Thus, the so-called mass scaling technique (Ref 13), which may be interpreted as the introduction of artificial inertia term, is usually employed in order to increase the size of time step. In these cases, the mass scaling factor must be selected carefully to get the solution with reasonable accuracy and simulation time. In sheet forming applications, the explicit method has been very successful.

**Rigid-plastic analysis** is more advantageous for computational efficiency and robustness than elasto-plastic analysis is. This method has
been used predominantly for the majority of bulk forming processes where the elastic deformation is negligible compared with the plastic deformation and the distribution of residual stresses is not of major concern. By neglecting the elastic portion of deformation, the rigid-plastic formulation (Ref 14) turns out to be very similar to those of fluid flow problems except for the presence of yielding and so it is sometimes called the flow formulation. The velocity field satisfying the equilibrium equations, constitutive equations, and boundary conditions instantaneously is obtained at each state in the course of deformation. Therefore, it is necessary to adopt an appropriate scheme for updating deformed configurations from the velocity field obtained at each state.

Unlike the elasto-plastic formulation, the rigid-plastic formulation does not have any ambiguity related to the choice of the objective stress rates and the decomposition of deformation gradient into the elastic and plastic parts (Ref 15, 16).

Updated Lagrangian (UL) Formulation.

There are several different formulations (Ref 17, 18) for continuum mechanics problems with large deformation and/or large rotation, e.g., total Lagrangian (TL), updated Lagrangian (UL), Eulerian, and arbitrary Lagrangian Eulerian (ALE) methods. In the first two methods, the motion of a continuum is described in terms of the coordinates of a material particle in an arbitrarily chosen reference configuration. The TL method uses the initial undeformed configuration as a fixed reference configuration, but the UL method uses the most updated configuration during the progress of deformation as the reference configuration. Both methods are amenable to solid mechanics problems in which the configuration of a boundary is not fixed in space and changes in the course of deformation. Compared with the TL method, the UL method can use the simplified kinematics of a continuum assuming that the reference configuration is updated continuously in a small amount of increment of deformation according to the desired degree of solution accuracy. Thus, the UL method has been used widely in most metal forming simulations. It is worthwhile to mention that this method generally requires frequent remeshing (or rezoning) when mesh is severely distorted in the process of large deformation. Furthermore, it may not be an effective method for a certain class of forming problems, such as extrusion, rolling, and machining, in which the steady state solution is of major concern.

In the Eulerian method, the motion of a continuum is described in terms of the spatial coordinates of a material particle in the current configuration. Therefore, this method is particularly suitable for well-constrained fluid flow problems in which the domain of interest is fixed in the space. However, there are inherent difficulties in applying this method to the moving boundary or free surface problems. The AL method has been developed to remove these difficulties by introducing the concept of mesh motion independent of the motion of material. The motion of a mesh can be chosen arbitrarily, but it must preserve the boundary of a continuum in the course of deformation. When the motion of mesh is set the same as the motion of material, it becomes the same as the Lagrangian methods. Several different schemes regarding the selection of mesh motion have been proposed according to the specific area of applications. The ALE method has been considered a prominent method, especially for the steady state metal forming applications. However, it has disadvantages, such as the increase of problem size due to additional variables for defining the mesh motion and the requirement of additional computation due to the evaluation of convection terms in updating the state variables.

Description of the Problem

The governing partial differential equations and the associated boundary conditions for the rigid-plastic and rigid-viscoplastic problems can be written as (Ref 12):

\[ \sigma_{ij} = \sigma'_{ij} + \sigma_m \] \hspace{1cm} (Eq 23)

\[ a_{ij} = 0 \quad \text{in } \Omega \] \hspace{1cm} (Eq 24)

\[ \sigma'_{ij} = \frac{2}{3} \epsilon_{ij} \] \hspace{1cm} (Eq 25)

\[ \dot{\epsilon} = \frac{1}{2} (u_{ij} + u_{ji}) \quad \text{in } \Omega \] \hspace{1cm} (Eq 26)

\[ \vec{a} = \vec{a} (\vec{e}, \vec{e}, T) \quad \text{in } \Omega \] \hspace{1cm} (Eq 27)

\[ u_i = u_i^0 \quad \text{on } \Gamma_u \] \hspace{1cm} (Eq 28)

\[ \mu \vec{t} = f_i \quad \text{on } \Gamma_f \] \hspace{1cm} (Eq 29)

Friction & contact conditions on \( \Gamma_c \)

Here, the open domain \( \Omega \) and its associated boundary \( \Gamma_c \) represent the current configuration of a body according to the UL formalism. The subscripts u, t, and c in \( \Gamma \) are used to denote three different types of the boundary associated with the boundary conditions: the prescribed velocity, the prescribed traction, and the frictional contact conditions, respectively. Indices i, j, and k are used to denote the components of a tensor and a comma denotes the spatial derivative with respect to the current configuration.

In Eq 23, i.e., the equilibrium equation, \( \sigma'_{ij} \) is the stress tensor, and \( \sigma_{ij} \) and \( \sigma_m \) denote its deviatoric and volumetric components, respectively, such that:

\[ \sigma'_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} \] \hspace{1cm} (Eq 31)

and

\[ \sigma_m = \frac{2}{3} \sigma_{kk} \] \hspace{1cm} (Eq 32)

where \( \delta_{ij} \) is the Kronecker delta, and the repeated index denotes the summation. It is noted that the stress measure in Eq 23 must be the first kind of Piola-Kirchhoff stress tensor according to the framework of the referential description of continuum mechanics. However, it can be assumed that the reference configuration is updated as frequently as required, so it is therefore not distinguished from the Cauchy stress tensor.

Equation 24 represents the incompressibility condition where \( u_i \) is the velocity vector. Equation 25 represents the constitutive equation based on the so-called \( J_2 \) flow rule. Here, \( \epsilon_{ij} \) is the strain-rate tensor or the symmetric part of the velocity gradient tensor as defined in Equation 26. \( \vec{a} \) and \( \vec{e} \) are the effective stress and the effective strain-rate, respectively, defined by:

\[ \vec{e} = \vec{e}^0 + \dot{\vec{e}} \Delta t \] \hspace{1cm} (Eq 33)

Equation 27 represents an implicit form of the yield criterion as a function of the effective strain \( \vec{e} \), the effective strain rate \( \dot{\vec{e}} \), and the temperature \( T \). Provided the increment of time between any two adjacent referential configurations is small enough, the effective strain can be evaluated approximately such that:

\[ \vec{e}_{\text{new}} = \vec{e}_{\text{old}} + \dot{\vec{e}} \Delta t \] \hspace{1cm} (Eq 34)

where \( \Delta t \) denotes the increment of time. Equation 27 represents the most popular form of the rigid-plastic constitutive equations, but more complicated forms of constitutive equations can, of course, be used within this structure of formulation such as the constitutive equations having internal state variables with the associated evolution equations.

Equations 28 and 29 are the prescribed velocity and traction boundary conditions, respectively. The hat symbol on \( n_t \) and \( t_s \) is used to denote prescribed values, and the vector \( n_t \) denotes the outward normal vector to the body at a point on the boundary. The friction and contact conditions are discussed in detail in the subsequent section.

Friction and Contact Conditions

The friction and contact conditions in this section are described between any two boundaries whether they belong to the same body (i.e., self-contact), rigid and deformable bodies, or two or more deformable bodies. Although several different methods (Ref 19–21) have been presented in describing the macroscopic contact and friction phenomena within the framework of continuum mechanics, the method of pointwise description of friction and contact conditions has been used in most practical applications. The contact condition and friction can be summarized as:

- Contact condition (non-penetration condition):
  - a. Any material particle of a given body cannot penetrate into another
  - b. The normal component of contact traction must be compressive for each body
Friction condition:

a. The magnitude of the tangential component of contact traction must be less than or equal to that of the normal component multiplied by a coefficient of friction.

b. The instantaneous relative motion in the tangential direction for a pair of contact points can take place when the equality in (a) above holds.

c. The tangential relative motion must be along the same line as the tangential component of contact traction but in the opposite direction.

In most metal forming applications, two different types of friction laws have been widely used: the Coulomb’s law of friction and the shear friction law. Statement (a) under “Friction condition” represents the Coulomb’s law of friction, but it also represents the shear friction law by replacing the normal component of contact traction by the shear yield stress of the weaker material. Also, the coefficient of friction is generally referred to as the constant factor in the shear friction law. This statement of contact and friction conditions can be expressed in a mathematical form as:

\[ \dot{z}_i = \lambda_{ni} \mathbf{r}_a \mathbf{n}_i + \dot{\lambda}_i (f(t) \frac{\left| \mathbf{p}_i \right|}{\left| \mathbf{p}_i \right|}) \]  
(Eq 35)

where \( \dot{z}_i \) is the relative velocity between two points in contact at an instant as:

\[ \dot{z}_i = \dot{a}_i - \dot{a}_i^T \]  
(Eq 36)

where the superscripts a and b denote the corresponding contact surfaces chosen arbitrarily, \( \mathbf{n}_i \) is the outward normal vector with respect to the contact surface “a”, \( \mathbf{p} \) is the contact traction vector on the contact surface “a” as:

\[ \mathbf{p}_i = \mathbf{p}_i - \mathbf{p}_i^T \]  
(Eq 37)

where the subscripts \( n \) and \( t \) denote the normal and tangential components, respectively, as:

\[ \dot{z}_i = \dot{a}_i \mathbf{n}_i + (\dot{\lambda}_i) \mathbf{p}_i \quad \text{and} \quad \mathbf{p}_i = \dot{a}_i \mathbf{n}_i + (\dot{\lambda}_i) \mathbf{p}_i \]  
(Eq 38)

Continuing with the description of terms in Eq 35, the symbol \( | \cdot | \) denotes the absolute magnitude of a vector, and \( \lambda_{ni} \) and \( \dot{\lambda}_i \) are non-negative constants. \( h(g) \) is an indicator function defined for any scalar-valued function \( g \) as:

\[ R(g) = \begin{cases} 1, & \text{if } g < 0 \\ 0, & \text{if } g \geq 0 \end{cases} \]  
(Eq 39)

In order to describe the friction laws effectively, a slip function \( f \) is introduced for each pair of contact points similar to the yield function in the classical theory of plasticity. The slip function is defined by:

\[ f(p) = |p| + \mu p \]  
(Eq 40)

and

\[ f(p) = |p| - mk \]  
(Eq 41)

corresponding to the Coulomb’s law of friction and the shear friction law, respectively, where \( \mu \) is the coefficient of friction, \( k \) is the shear yield stress of the weaker material, and \( m \) is a constant friction factor.

It is noted that the frictional dissipation function associated with the friction law in Eq 38 or 39 has a non-differentiable form with respect to its primary variables, i.e., the velocities of each body. In the area of metal forming simulation, the following form of regularized friction laws has been widely used. For any \( \zeta_i \):

\[ \mathbf{p}_i = \mu \mathbf{p}_i \left[ \frac{\tan \left( \frac{\zeta_i}{\zeta_i^*} \right)}{\sec \left( \frac{\zeta_i}{\zeta_i^*} \right)} \right] \]  
(Eq 42)

and

\[ \mathbf{p}_i = -m k \left[ \frac{\tan \left( \frac{\zeta_i}{\zeta_i^*} \right)}{\sec \left( \frac{\zeta_i}{\zeta_i^*} \right)} \right] \]  
(Eq 43)

where \( \zeta_i^* \) is a positive constant. These regularized friction models approach the original friction laws asymptotically as \( \zeta_i \) approaches zero. However, a very small value of \( \zeta_i^* \) can make a difficulty in convergence, and a larger value may give a solution deviated from the original friction laws.

**Mixed Variational Formulation**

Boundary value problems (BVP) in continuum mechanics can be expressed in two different ways: the partial differential equations (PDE) with the associated boundary conditions and the variational equations with the appropriate function space. The solutions of these two different forms of BVP are referred to as the strong solution and the weak solution, respectively. The term weak is used in the sense that the requirement of continuity (differentiability) of solution is weakened in the variational form of BVP. If a strong solution requires the existence of a second derivative, the corresponding weak solution requires only the existence of a first derivative in the sense of distribution; that is, the first derivative needs to be continuous within each of a finite number of subdomains but not necessarily across the interboundary between subdomains.

The variational form of BVP can be obtained only when the quadratic form of functional exists so that the set of Euler equations, obtained by the vanishing of the first variation of the functional, is identical to the original PDE. However, the same form can be obtained by using the so-called weak formulation or the principle of virtual work, although certain mathematical features of variational problems with the quadratic functional such as the existence and uniqueness of solutions and the stability and accuracy of finite element solutions cannot be stated. Here, the weak formulation is used to accommodate a broad class of plastic constitutive models to the same framework of formulation.

The constraint conditions such as the incompressibility condition and the contact condition can generally be incorporated into the variational formulation by using one of two techniques: the penalty method or the Lagrange multiplier method. The penalty method has the advantage of simple implementation, but it has a drawback such that it can result in an overconstrained problem or an underconstrained problem depending on the choice of the penalty parameter. The overconstraint means the volumetric locking or the locking of contact surfaces, and the underconstraint means the inaccuracy of solution in the sense of incompressibility or nonpenetration. On the other hand, the Lagrange multiplier method can avoid the drawback of the penalty method, but it has a disadvantage concerning the increase of problem size because the Lagrange multipliers are treated as additional solution variables such as the velocity of material particles. The Lagrange multiplier can be interpreted as the hydrostatic stress for the incompressibility constraint and the normal contact traction for the nonpenetration condition.

In areas of metal forming simulation, it is popular to use the Lagrange multiplier method for the incompressibility condition and the penalty method for the contact condition. However, the penalty method has also been used successfully with a certain class of finite elements with the selective reduced integration scheme.

**Finite Element Formulation**

The finite element method can be distinguished from other approximate methods by the way it constructs the trial solution (e.g., kinematically admissible velocity field) with a finite number of piecewise continuous trial functions (polynomial functions in most applications) (Ref 22–24).

For a particular class of metal forming processes, it is always important to select an appropriate type of element (i.e., the order of polynomials, the geometric shape, the rule of numerical integration, and so on). Some issues related to the selection of element are discussed subsequently. From the standpoint of polynomial order, linear elements are generally preferred to quadratic or higher-order elements for most metal forming applications in which the friction and contact conditions are always present. With use of the friction and contact conditions described previously, the so-called node-to-segment contact situation cannot be avoided because these constraints need to be imposed point-wise. The node-to-segment contact can be treated in a simpler manner with linear elements rather than higher-
order elements. Also, linear elements are generally easier to use without a-priori knowledge of solution than higher-order elements with the same degrees of freedom for a given problem.

Next, from the standpoint of geometric shape, there are two different kinds of linear elements: triangular or quadrilateral for two-dimensional elements and tetrahedral or hexahedral for three-dimensional elements. Shape functions for triangular and tetrahedral elements contain polynomial terms such as \((1, \xi, \eta)\) and \((1, \xi, \eta, \zeta)\) respectively. On the other hand, those for quadrilateral and hexahedral elements contain polynomial terms such as \((1, \xi, \eta, \zeta)\) and \((1, \xi, \eta, \zeta, \xi_1, \eta_1, \zeta_1)\) respectively. Triangular and tetrahedral elements are known as constant stress/strain elements (CST) because all derivatives with respect to any component of coordinates vanish. In the rigid-plastic formulation, vanishing derivatives imply constant strain rates and, therefore, constant stresses, within an element. On the other hand, the quadrilateral and hexahedral elements have possible derivatives with respect to coordinates, and thus, the velocity gradient in one component of coordinates is linear with respect to the other components of coordinates. Strain rates and stresses are linear within an element accordingly. Both kinds of linear elements have been used widely by weighing the pros and cons for a particular application.

Generally speaking, triangular and tetrahedral elements have more flexibility in filling meshes into any complicated shape than quadrilateral and hexahedral elements. It is noted that triangular and tetrahedral elements in the group of CST elements must be distinguished from the so-called degenerated elements whose strain rates and stresses are not constant within an element.

Degenerated elements can be obtained from coalescing adjacent nodes into the same node, for instance, mapping from the four-node quadrilateral parent element into a three-node triangular shape or mapping from the eight-node hexahedral parent element into a four-node tetrahedral shape. The performance of degenerated elements is not as good as that of the CST elements or the original quadrilateral and hexahedral elements. Also, a numerical integration scheme for the CST elements in evaluating the stiffness matrix is unnecessary because they have constant values of strain rate and stresses within an element. In other words, the integration is always exact since the integrand is constant with respect to coordinates. On the other hand, for quadrilateral and hexahedral elements, a numerical integration rule based mostly on the Gaussian quadrature formula is required in evaluating the stiffness matrix. The numerical integration is exact when a real element keeps the same shape as the parent element, i.e., a rectangular shape or a brick shape, although it is not possible except for problems with a very simple geometry. In such cases, the determinant of the Jacobian matrix mapping between the domains of the parent element and a real element is constant. Errors originated from the numerical integration increase as the shape of element is apart from that of the parent element.

Some disadvantages of CST elements are listed as follows. The CST elements are apt to have directionality in the mesh topology and thus in the solution because the mesh topology directly reflects on the form of trial solutions. Here, the directionality implies that triangular and tetrahedral elements are globally biased in a certain direction. Also, the CST elements may show more tendency of volumetric locking due to the incompressibility constraint condition than quadrilateral and hexahedral elements. The term volumetric locking means that each element endowed with a linear velocity field cannot deform properly when the same degree of satisfaction is required elementwise for both the deformation and the constraint condition. In order to avoid such a locking phenomenon, the degree of satisfaction for the constraint condition needs to be relaxed in a certain manner such as with the selective reduced integration scheme or with the mixed formulation as described previously. Another important aspect to be considered in degenerated elements is the penalty method for large deformation problems, may be whether an efficient and robust method for automatic remeshing is available with the shape of element. In areas of metal forming simulation, quadrilateral elements are preferred to triangular elements for the two-dimensional analysis, but tetrahedral elements have been used more than hexahedral elements for three-dimensional analysis because of the versatility in automatic remeshing.

As mentioned previously, linear tetrahedral elements cannot be used for metal forming simulation without an appropriate resolution of the volumetric locking problem. The most common approach to enforce incompressibility includes the penalty method and the Lagrange multiplier method as discussed previously. These methods, however, are limited to the quadrilateral element in two-dimensional and hexahedral element in three-dimensional. Furthermore, the resulting matrix is ill conditioned for a conjugate-gradient solver due to large penalty values for incompressibility. The mixed formulation uses the Lagrange multiplier method to achieve incompressibility condition. Generally speaking, the order of polynomial for the velocity shape function needs to be higher than that for the mean stress shape function because the mean stress is originally a dual variable paired with the volumetric strain rate that is the trace of the velocity gradient. For instance, when the linear shape function is used for both variables, the volumetric strain rate becomes constant within an element, but the mean stress is linear. It is obvious that a linear function cannot be matched with a constant value in a general case and a mesh system may tend to lock as a result. When the quadratic shape function is used for the velocity in order to make the well-posed problem, a considerable increase of problem size cannot be avoided.

The MINI element, the shape function for the velocity is enriched with bubble terms associated with an additional bubble node, although the shape function for the mean stress remains as a linear function interpolated with the values at four vertex nodes. The bubble node is located at the centroid of tetrahedron and has only the velocity degree of freedom. Here, the word bubble means that its value always vanishes along the boundary of the element. This element does not have the volumetric locking problem. Moreover, the total number of equations can be maintained as the same as that with the tetrahedral elements because the velocity components at the bubble nodes can be eliminated at the element level by means of the static condensation.

Although the bubble node is introduced mainly for a systematic stabilization of the ill-posed finite element system, a considerable amount of computational effort is additionally required for the static condensation and recovery of the velocity of the bubble node. However, such additional computational cost can be minimized with a few modifications of the standard MINI element. In fact, it turns out that the derivation of this modified MINI element can be conceived as a systematic stabilization of the standard mixed tetrahedral element because the effect of the bubble node appears on only the block diagonal terms associated with the mean stress.

Heat Transfer

In the metal forming industry, heating is frequently used to increase material workability and control forming loads. During the forming process, heat is generated from plastic and friction work. Heat is also lost through contact with colder dies and through convection and radiation with the environment. Since materials properties vary considerably with temperature, accurate temperature prediction is required. For convenience, the deformation analysis and thermal analysis can be loosely coupled in such a way that plastic work and friction are considered as heat source in the thermal analysis while the updated temperature field is used to determine the flow stress behavior during the deformation analysis.

The governing equation for heat transfer can be expressed as:

\[
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \frac{q}{\rho C} \quad (\text{Eq 44})
\]

where \(\rho\) is the density, \(C\) is the specific heat capacity, \(T\) is the temperature, \(t\) is time, \(k\) is the thermal conductivity, and \(q\) is a heat generation term. Heat generation in metal forming is due to work of plastic deformation and friction. Heat generation due to plastic deformation is given by

\[
\dot{q}_{\text{plastic}} = k \oint \mathbf{\dot{\varepsilon}} \cdot d\mathbf{V} \quad (\text{Eq 45})
\]

where \(k\) is a deformation efficiency term, representing the fraction of the work of deformation converted to heat.
The boundary condition for the tool-workpiece contact surface includes friction heating and heat exchange via temperature difference of two objects.

\[
\dot{q}_1 = \int_{S_1} \eta_1 \dot{V} \cdot \vec{n} \, dS_i + \int_{S_1} \dot{H} \, dS_i
\]

(Eq 46)

where \( \eta_1 \) is the friction coefficient, \( \dot{V} \) is the relative velocity of nodes, \( \vec{n} \) is the normal vector, and \( \dot{H} \) is the lubricant heat transfer coefficient, and \( \Delta T \) is the temperature difference between two objects.

The boundary condition of the free surface includes convection heat and radiation heat from/to the environment:

\[
\dot{q}_2 = \int_{S_2} \kappa \dot{T} \, dS_i + \int_{S_2} \sigma(T^n - T_s^n) \, dS_i
\]

(Eq 47)

where \( \kappa \) is the convection heat transfer coefficient, \( T_s \) is the environment temperature, \( \sigma \) is the Stefan-Boltzmann radiation constant, and \( \epsilon \) is the emissivity of the surface. It is also noted that the inclusion of the view factor for radiation heat calculation (which is not addressed here) can improve the accuracy of the thermal model, especially for the hot forging condition.

By substituting Eq 45 into Eq 44 and introducing a small, arbitrary variation \( \delta T \), and applying the divergence theorem, Eq 44 can be written in the form:

\[
\int_V \kappa \delta T \, dV + \int_V \dot{\sigma} \dot{T} \, dV + \int_{S_1} \kappa \dot{T} \, dS_i - \int_{S_2} \sigma(T^n - T_s^n) \, dS_i = 0
\]

(Eq 48)

where \( q_{sb} \) is the heat flux across the boundary. It includes the convection heat and radiation heat to the environment for the free surface and friction heat and heat gain or loss to the contacting surface. The temperature distribution function can be expressed through nodal temperatures and shape functions (Ref 12).

After discretization, Eq 48 can be further expressed in matrix form (Ref 24) as:

\[
CT + KT = Q
\]

(Eq 49)

where \( C \) is the heat capacity matrix, \( K \) is the heat conduction matrix, \( T \) is the nodal point temperature vector, and \( Q \) is the vector containing the time rate of change of temperature of node points. The heat flux vector, \( Q \), for metal forming simulations considers plastic work of deformation, heat generation due to sliding contact friction, and heat flux due to lubricant convection, convection, and radiation.

Example Simulations

To demonstrate the capability of the modeling techniques, four examples are included in the following section. The first example is the ring compression, and three methods (slab, UBM, and FEM) are used to study the process. The second example is the cold forging of an electrode, the third example is the ring forging of a crankshaft, and the last example considers material cutting. Only FEM is used in the last three examples due to the complexity of the processes.

Ring Compression

The ring compression test is one of the methods used to determine the friction factor. The process is generally assumed to be two-dimensional or axisymmetric. Due to the simplicity of the workpiece and die geometry, ring compression is used as the first example to demonstrate and compare the capability of slab, UBM, and FEM.

Two-ring test FEM simulations with outer diameter (OD), inner diameter (ID) and height ratios of 6:3:2 and 6:3:0.5 were carried out with a constant friction factor of 0.4. A constant flow stress of 70 MPa (10 ksi) was used in the simulation. The predicted shapes using the FEM and the associated neutral line within the workpiece at different stages of deformation in both cases are shown in Fig. 2 and 3, respectively. In the case with the ratio 6:3:2, the neutral line within the workpiece is a function of height. In the case with 6:3:0.5, the neutral line remains to be a vertical line. Furthermore, the ID and OD surface bulged due to the friction force on the two contacting surfaces.

To simplify the mathematical derivation, the assumed velocity field (Eq 12, 13) in the UBM is based on a rectangular workpiece shape. The radial velocity is a function of the radius and the neutral radius, the axial velocity is a function of the height position, and the normal velocity is a function of friction factor and shear stress.

This simplification/assumption introduces error when the ID and OD surfaces of workpiece start to bulge during the compression. As the workpiece height increases, it is known that the unstable flow (or buckling mode) will gradually dominate the deformation. In such situations, the assumed UBM velocity field is moving away from the true velocity field. To demonstrate the buckling mode, another ring compression example with OD, ID, and height ratio of 6:3:6 was carried out. In this case, the flow stress is assumed to be in the form of \( \sigma = 10 \epsilon^{0.1} \). A friction factor of 0.2 was used. The predicted FEM mesh and various deformation stages are shown in Fig. 4.

Figure 5 shows the effect of friction on the location of a neutral point with three different aspect ratios of a ring: i.e., 6:3:0.5, 6:3:1, and 6:3:2, in terms of the OD, the ID, and the H, in sequence. From this figure, it is noted that the upper-bound solutions approach the mean radius of ring asymptotically as the magnitude of shear friction factor increases. It is because, except for point sticking, neither local nor global sticking phenomenon can be represented with the present velocity field. It is also shown that the upper-bound solutions correlate better with the FEM solutions as the height of ring reduces.
extrusion operations were modeled in two dimensions, and the final forming was modeled in three dimensions. The shearing and squaring operations were not modeled in this study.

Two laps were developed during the backward extrusion as shown in Fig. 8(a) and (b). One lap is seen on the outside wall (Fig. 8b) and extends the entire circumference. The other lap occurred only partially on the inside wall (Fig. 8a). It is believed that eccentricity of the forming process caused the internal lap. From visual inspection, the lap occurred in every tooth in the final part as shown in Fig. 8(c). Figure 8(c) also shows a change in the texture at the inner wall of the electrode, above the teeth. Figure 8(d) shows a lap that occurred at the bottom of the cavity, which is seen as a round pattern pointed to by the arrow. Due to the size of the part, it is very difficult to visualize the fold, other than the one on the tooth tip.

The strain distribution and predicted geometry at the end of each operation through extrapolation are shown in Fig. 9. It is seen that the external fold is successfully predicted in the zoomed area of the figure. Due to the axisymmetric assumption, the internal fold is not seen in the simulation results. Further study, considering tooling eccentricity, will be carried out in the near future. Due to the symmetry condition, 1/12 of the workpiece was modeled in the simulation to reduce the simulation time.

The predicted geometry at early stage of the tooth forming is shown in Fig. 10. It is clearly seen that the material is pushed both upward and downward through the contact of the punch. Folds occurred on the wall above the teeth and below the teeth. It is also noted that the partially formed tooth is seen as a concave shape. As the punch moved farther down, the fold began to wrap around the top of the teeth (Fig. 11). Figure 12 shows that this material peeled down into the bottom of the cavity as shown in Fig. 8(d). Also, the fold above the teeth is shown to smear, which correlates well to Fig. 8(c) as the cause of the uneven texture. The final modeled part is seen in Fig. 13. As shown in the real part, the tooth tip is the last area to fill. A clear fold is predicted in the same area.
The flow pattern and potential defects such as under fill or folding, the forging load for tool stress analysis and press selection, stress and strain distribution for possible evaluation of microstructure, improvement and optimization of the existing forging processes are important for an accurate simulation. Further, the flow stress is generally sensitive to the strain rate at a hot forging condition; ram speed will also play an important role, not only to predict accurately in flow pattern, but also in the prediction of the load. Interface properties such as friction and heat transfer coefficient are also important variables that will influence the heat loss rate of the workpiece, material flow, and forging load. Therefore, an accurate model should take into account the coupling between the process variables, for example, ram speed, friction factor, heat transfer coefficient, and the material data (e.g., flow stress representation and thermal data).

Toward this goal, a parametric study was first carried out to evaluate these critical process variables to assure the accuracy of the model. The variables under study included the flow stress, friction factor, heat transfer coefficient, and billet temperature.

The materials properties for AISI-1045 and AISI-1055 were selected for the simulation. There are three stages to forge the crankshaft: busting, blocking, and finishing.

The workpiece is first heated to 1200°C (2192 °F). All dies are heated to 200°C (392 °F). As shown in Fig. 14, after 10 s of air transfer, the workpiece is placed in the buster dies for the first forging operation. After 1.5 s air transfer, the workpiece is removed from buster and placed in the blocker for the second forging process. After another 1.5 s air transfer, the workpiece is removed from blocker and placed in the finisher for the final forging process. A 6500 ton mechanical press is used in the forging and simulation. To accurately model the temperature evolution during the entire forging process, the simulation was carried out in a nonisothermal manner, including heat transfer analysis to account for the air transfer time.

During the parametric study, it was found that an accurate lubricant heat transfer coefficient was very critical to obtain a similar flash profile as the actual part. The actual workpiece at the end of the blocker operation, the predicted shape and the actual shape are in excellent agreement as shown in Fig. 17. It is also noted that the folding defects (also known as laps) are successfully predicted by the simulation as shown in Fig. 18 and 19. The prediction of the lap shape and the propagation pattern are well correlated with the corresponding experimental results.

To validate the model further, the flash thickness and the corresponding forging load are measured and compared with the numerical prediction with excellent agreement as shown in Fig. 20.

**Cutting Modeling**

Machinability is of primary interest in the materials cutting process. It is affected by many
factors, among which are materials behaviors, insert shape, and cutting condition. A good understanding of the interactions among the chip flow, heat generation, residual stress, tool stress, and tool wear is crucial in order to optimize the design of the process and tooling (Ref 27).

To understand the thermal-mechanical response on an insert (tool wear) during a long period of cutting, the conventional updated Lagrangian transient approach, especially for three dimensional analysis, is not efficient since it requires enormous CPU time to reach a steady state. A hybrid procedure using both UL formulation for transient analysis and Eulerian formulation for steady state has been developed (Ref 27). In this hybrid procedure, the transient UL method is first used to predict the initial chip formation as shown in Fig. 21. The solution at the end of the transient run is then used as an initial guess for the steady state analysis.

In most solid mechanics problems, the domain, i.e., the steady state configuration of a deformable body, is unknown a priori and must be determined as a part of the solution. From the standpoint of the FEM, each node has degrees of freedom for both the velocity and the position. Thus, the velocity field needs to be satisfied for the equilibrium equations, the constitutive equations and the boundary conditions, and the configuration needs to be satisfied for the free surface condition and the contact condition. Here, the free surface condition implies that the velocity on the free surface must be tangent to the surface and there is no traction on the surface.

A great number of different solution methods have been presented, including ALE methods, pseudo-solid domain mapping methods, adaptive h-p FE methods, streamline tracking methods, and methods of splines. In this current approach, two sets of coupled governing equations are solved for both the velocity and the position iteratively. First, the Eulerian velocity solution is obtained based on a given configuration. Next, the new configuration, i.e., the nodal position, is updated so that it satisfies the free surface condition and contact condition with the given velocity field. The first part is based on the standard flow formulation of plastic materials. For the sake of computational effectiveness, the second part is divided into two levels—that is, the determination of the position of surface nodes and internal nodes.

Employing surface elements that have both membrane and edge bending stiffness then solves the position of surface nodes. The free surface condition is also embedded into the elemental equations so that each free surface element is positioned to be parallel to the velocity at the center of element. Since the resulting set of equations is nonlinear due to the geometric nonlinearity, the Newton-Raphson iterative method is used. Once the position of surface nodes is obtained, employing incompressible elastic solid elements solves for the position of internal nodes.

State variables such as stresses and strains are updated in two different ways: simple interpolation scheme of existing solutions and streamline tracking scheme, depending on whether the updated Lagrangian solutions are available. A result of steady state thermomechanical machining simulation is presented in Fig. 22. The steady state insert temperature is shown in Fig. 23. The work to validate the cutting force, temperature, and wear using the model is being carried out (Ref 27 and 28).